Some Notes on Luminosity Calculations

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Abstract

Recent interest in possible effects on luminosity due to crossing angles, transverse offsets, and "hour glass" effects has generated a lot of discussion. In this note I wish to show the general approach to estimating the magnitude of these effects and give some practical results.

1 General Formalism

I'll assume that the beams are Gaussian in all three degrees of freedom, and will implement a "paraxial approximation" when referring to crossing angles (i.e., the angles are small with respect to the ideal trajectory). I will also ignore effects of transverse coupling. In addition, we will assume that the two beams have equal transverse (95% normalized) emittances ($\epsilon_{x,y} = \epsilon$ for beams #1 and #2) and equal bunch lengths to simplify the discussion. It should be straightforward to see how different transverse and longitudinal emittances could be incorporated into the discussion if desired.

For the case of the Tevatron, the separated orbits are generated by electrostatic fields and hence, relative to the "central orbit" (orbit with separators off) the proton and antiproton trajectories are reflections of each other. Therefore, we take the "central orbit" as our reference coordinate, z, with the collision point in the absence of separators at $z = 0$. The horizontal and vertical components of the separated trajectories across an interaction region are opposite for the two beams [$x_1(z) = -x_2(z)$ and $y_1(z) = -y_2(z)$]. Now imagine when viewed from above, the horizontal trajectory of "beam #1" is displaced from the z-axis at the collision point by a distance $\Delta_x/2$, and has slope $\alpha/2$ as depicted in Figure 1. Likewise, for "beam #2," the trajectory is displaced by $-\Delta_x/2$ and has slope $-\alpha/2$. A similar picture can be drawn for the vertical trajectories when viewed from the side. We take $t = 0$ to be the time at which the bunches are at longitudinal coordinate $z = 0$.

The interaction rate produced by the bunches passing through each other can be given by the following classical argument. Suppose the particles passing by each other have an interaction cross section Σ_{int}. Then the probability of a single particle from beam #1 interacting with a particle in beam #2 while in a particular volume $dxdydz$ is $\Sigma_{int}/(dxdy)$ times the number of particles within the volume from beam #2, $\rho_2(x, y, z, t)dxdydz$. Then multiply by the number of particles in the

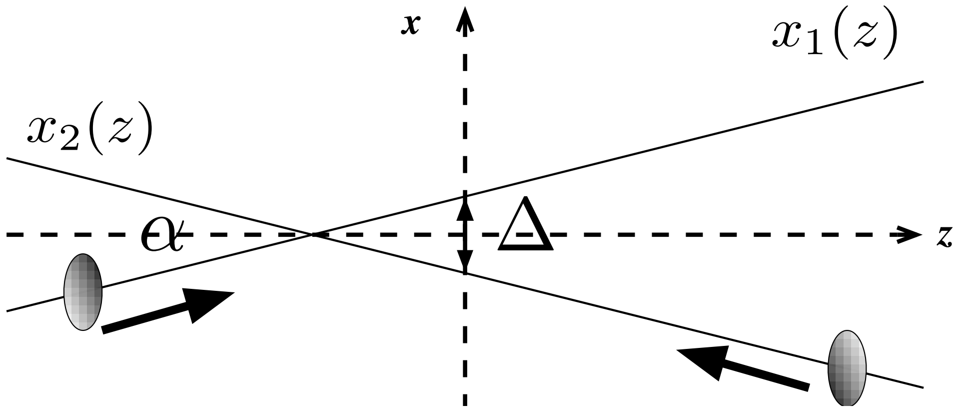

Figure 1: Beam trajectories in the z-x plane. The y coordinate is directed out of the page. A similar view exists in the z-y plane.

volume from beam #1, $\rho_1(x, y, z, t)dxdydz$ to get the total probability of an interaction. To get a rate, we multiply by the rate at which particles enter the volume from the left and from the right, which is $v_1/dz - v_2/dz$. Then, we integrate along t to get the total number of collisions during the entire time the bunches pass through each other. If bunches "collide" with frequency f, then the average interaction rate at the collision point would be

$$\mathcal{R} = f \int_{x,y,z} \int_t \left(\frac{\Sigma_{int}}{dxdy} \right) (\rho_2 dxdydz)(\rho_1 dxdydz) \frac{(v_1 - v_2)}{dz} dt$$

Since the bunches are approaching each other, $v_1 = -v_2 = v \approx c$.

Factoring out the interaction cross section leaves the "luminosity," L:

$$L = 2cf \int_{x,y,z} \int_t \rho_1(x, y, z, t)\rho_2(x, y, z, t)dxdydzdt.$$

1.1 Particle Distribution Functions

According to our Gaussian assumption, we would write the particle density function as

$$\rho = \frac{N}{(\sqrt{2\pi})^3 \sigma_x \sigma_y \sigma_z} \; e^{-(x-\mu_x)^2/2\sigma_x^2} \;\; e^{-(y-\mu_y)^2/2\sigma_y^2} \;\; e^{-(z-\mu_z)^2/2\sigma_z^2}.$$

For the transverse coordinates, the centers move along the two trajectories, which are functions of z, and their transverse widths will also vary with z according to the amplitude function of the accelerator. In z, the location of the beam center will vary with time. So, we can write

$$\begin{aligned}
\rho_1 &= \frac{N_1}{(\sqrt{2\pi})^3 \sigma_x(z)\sigma_y(z)\sigma_z} \; e^{-(x-x_1(z))^2/2\sigma_x^2(z)} \;\; e^{-(y-y_1(z))^2/2\sigma_y^2(z)} \;\; e^{-(z-ct)^2/2\sigma_z^2} \\
\rho_2 &= \frac{N_2}{(\sqrt{2\pi})^3 \sigma_x(z)\sigma_y(z)\sigma_z} \; e^{-(x-x_2(z))^2/2\sigma_x^2(z)} \;\; e^{-(y-y_2(z))^2/2\sigma_y^2(z)} \;\; e^{-(z+ct)^2/2\sigma_z^2}
\end{aligned}$$

1.2 Transverse Position and Transverse Widths along the Trajectory

With our coordinates, in accordance with Figure 1, the two trajectories relative to the central orbit and the beam widths in the horizontal and vertical dimensions are given by

$$x_1(z) = \frac{1}{2}(\Delta_x + \alpha_x z) \qquad x_2(z) = -\frac{1}{2}(\Delta_x + \alpha_x z)$$
$$y_1(z) = \frac{1}{2}(\Delta_y + \alpha_y z) \qquad y_2(z) = -\frac{1}{2}(\Delta_y + \alpha_y z)$$

$$\begin{aligned}
\sigma_x^2(z) &= \frac{\epsilon\beta_x^*}{6\pi\gamma}\left[1 + \left(\frac{z - z_{x0}}{\beta_x^*}\right)^2\right] \\
\sigma_y^2(z) &= \frac{\epsilon\beta_y^*}{6\pi\gamma}\left[1 + \left(\frac{z - z_{y0}}{\beta_y^*}\right)^2\right].
\end{aligned}$$

Here, we acknowledge that the amplitude functions β_x and β_y are quadratic functions of z, with minima (β_x^*, β_y^*) at locations z_{x0} and z_{y0}, respectively. Ideally, the Tevatron would be operating with $\beta_x^* = \beta_y^* = \beta^* = 35$ cm, and with $z_{x0} = z_{y0} = 0$.

1.3 Putting it Together ...

... we get

$$\begin{aligned}
L &= \frac{2cfN_1N_2}{(2\pi)^3\sigma_z^2}\int_{x,y,z}\int_t \frac{1}{\sigma_x^2(z)\sigma_y^2(z)} e^{-(2x^2+x_1^2+x_2^2)/2\sigma_x^2(z)} \\
&\quad \times \; e^{-(2y^2+y_1^2+y_2^2)/2\sigma_y^2(z)} \; \times \; e^{-(2z^2+2c^2t^2)/2\sigma_z^2} \; dt\, dxdydz \\
&= \frac{2cfN_1N_2}{(2\pi)^3\sigma_z^2}\int_{-\infty}^{\infty} e^{-c^2t^2/\sigma_z^2} dt \\
&\quad \times \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{1}{\sigma_x} e^{-x^2/\sigma_x^2} dx \int_{-\infty}^{\infty} \frac{1}{\sigma_y} e^{-y^2/\sigma_y^2} dy \right. \\
&\qquad\qquad \left. \times \; \frac{1}{\sigma_x\sigma_y} \; e^{-(x_1^2+x_2^2)/2\sigma_x^2} \; e^{-(y_1^2+y_2^2)/2\sigma_y^2} \; e^{-z^2/\sigma_z^2} \right\} dz \\
&= \frac{2cfN_1N_2}{(2\pi)^3\sigma_z^2}\left(\frac{\sqrt{\pi}\sigma_z}{c}\right)(\sqrt{\pi})(\sqrt{\pi})\int_{-\infty}^{\infty} \frac{1}{\sigma_x\sigma_y} \; e^{-(x_1^2+x_2^2)/2\sigma_x^2} \; e^{-(y_1^2+y_2^2)/2\sigma_y^2} \; e^{-z^2/\sigma_z^2} \; dz.
\end{aligned}$$

Now

$$x_1^2 + x_2^2 = \frac{1}{2}(\Delta_x + \alpha_x z)^2$$

and likewise for y, and in addition

$$\sigma_x\sigma_y(z) = \frac{\epsilon}{6\pi\gamma}\sqrt{\beta_x^*\beta_y^*}\sqrt{\left[1 + \left(\frac{z - z_{x0}}{\beta_x^*}\right)^2\right]\left[1 + \left(\frac{z - z_{y0}}{\beta_y^*}\right)^2\right]}.$$

Thus we can arrive at

$$L = \frac{fN_1N_2}{4\pi\sigma_0^2} \cdot \left(\frac{\beta^*}{\sqrt{\beta_x^*\beta_y^*}}\right) \frac{1}{\sqrt{\pi}\sigma_z} \int_{-\infty}^{\infty} \frac{\exp\left\{-\left[\frac{\Delta_x+\alpha_x z}{2\sigma_x(z)}\right]^2 + \left[\frac{\Delta_y+\alpha_y z}{2\sigma_y(z)}\right]^2\right\}}{\sqrt{\left[1+\left(\frac{z-z_{x0}}{\beta_x^*}\right)^2\right]\left[1+\left(\frac{z-z_{y0}}{\beta_y^*}\right)^2\right]}} \; e^{-z^2/\sigma_z^2}\, dz \qquad (1)$$

where β^* is the design value of β at the interaction point, and σ_0 is the design value of the rms beam size, given from $\sigma_0^2 = \epsilon\beta^*/(6\pi\gamma)$.

2 Reduced Expressions

Let's look at some simple cases ...

2.1 Short Bunches

Suppose we have the ideal case where the amplitude functions have design values, and the trajectories are exactly head-on. And suppose that the bunch lengths are short enough that the amplitude function does not vary over the length of the luminous region ($\sigma_z \ll \beta^*$). Then, we are left with evaluating

$$\frac{1}{\sqrt{\pi}\sigma_z}\int_{-\infty}^{\infty} e^{-z^2/\sigma_z^2}\, dz = \frac{1}{\sqrt{2\pi}\sigma_z}\int_{-\infty}^{\infty} e^{-u^2/2\sigma_z^2}\, du = 1$$

and we have

$$L = \frac{fN_1N_2}{4\pi\sigma_0^2} \equiv L_0$$

We note that along the z coordinate,

$$\frac{dL}{dz} = L_0 \cdot \frac{1}{\sqrt{2\pi}(\sigma_z/\sqrt{2})} e^{-z^2/(\sigma_z/\sqrt{2})^2}$$

and thus we see that the luminous region has rms extent $z_{rms} = \sigma_z/\sqrt{2}$.

2.2 Long Bunches

The Tevatron has bunch lengths which are greater than β^*, and hence the luminous region is spread out and the average luminosity is reduced. For $\sigma_z = 2\beta^*$, for instance, then at $z = \sigma_z/\sqrt{2}$ the amplitude function would have a value $\beta^*(1 + (\sqrt{2})^2) = 3\beta^*$. Thus, at $\pm\, z_{rms}$ the beam is about $\sqrt{3} = 1.7$ times larger than at the midpoint of the luminous region.

For long bunches but with otherwise ideal parameters, Equation 1 reduces to

$$L = L_0 \cdot \mathcal{H} = L_0 \cdot \frac{1}{\sqrt{\pi}\sigma_z} \int_{-\infty}^{\infty} \frac{e^{-z^2/\sigma_z^2}}{1 + \left(\frac{z}{\beta^*}\right)^2} \, dz$$

The integral on the right, $\mathcal{H}$, is called the “hour glass factor,” and can be evaluated in terms of the Error Function,

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

which can be looked up in standard tables, and is often provided numerically in standard software packages. The luminosity result is,

$$L = L_0 \cdot \mathcal{H}(\beta^*/\sigma_z) = L_0 \cdot \sqrt{\pi} \left(\frac{\beta^*}{\sigma_z}\right) e^{(\beta^*/\sigma_z)^2} \left[1 - \mathrm{erf}(\beta^*/\sigma_z)\right].$$

The hour glass factor as a function of β^*/σ_z is plotted in Figure 2. Note that the luminous region

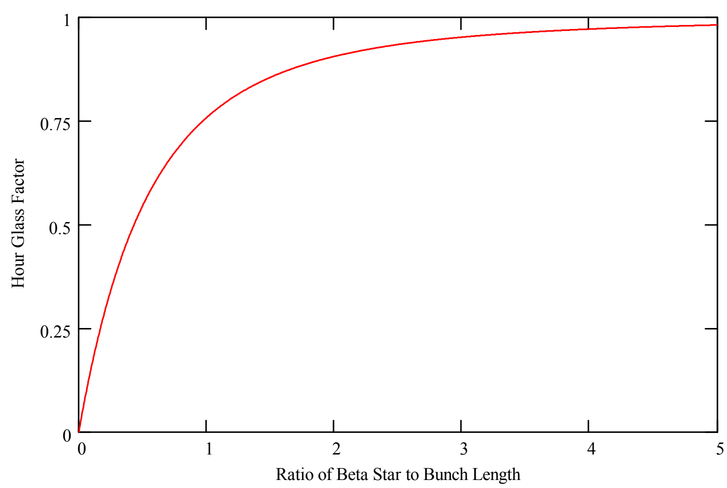

Figure 2: Luminosity as function of ratio of β^* to σ_z.

is no longer Gaussian, but its rms extent can still be evaluated analytically. The result, with $u \equiv \beta^*/\sigma_z$, is

$$z_{rms}(u) = \beta^* \left\{ \frac{\frac{1}{u} e^{-u^2}}{\sqrt{\pi}[1 - \mathrm{erf}(u)]} - 1 \right\}^{1/2}.$$

Taking the limit as $u \to \infty$ (bunches become short), we easily see that the result of the previous section emerges:

$$\begin{aligned}
\lim_{u\to\infty} \langle z^2 \rangle &= \beta^{*2} \lim_{u\to\infty} \left\{ \frac{\frac{1}{u} e^{-u^2}}{\sqrt{\pi}[1 - \mathrm{erf}(u)]} - 1 \right\} \\
&= \beta^{*2} \lim_{u\to\infty} \left\{ \frac{-\frac{1}{u^2} e^{-u^2} - 2e^{-u^2}}{\sqrt{\pi}\left(-\frac{2}{\sqrt{\pi}} e^{-u^2}\right)} - 1 \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \beta^{*2}\left\{\frac{1}{2u^2}+1-1\right\}\\
&= \beta^{*2}/2u^2\\
&= \sigma_z^2/2.
\end{aligned}
$$

2.3 Offset Centers

Next, suppose once again that the bunches are short, but allow the two opposing beams to be offset in one degree of freedom by a distance Δ, though remain parallel to each other ($\alpha = 0$). Then, Equation 1 reduces to

$$L = L_0 \cdot \frac{1}{\sqrt{\pi}\sigma_z}\int_{-\infty}^{\infty} e^{-\Delta^2/4\sigma_0^2} e^{-z^2/\sigma_z^2}\, dz = L_0 \cdot e^{-(\Delta/2\sigma_0)^2}.$$

The luminosity is reduced, but the length of the luminous region is not affected.

2.4 Crossing Angle

Here, we assume the centers of the bunches pass through each other at the desired interaction point, but do so with a crossing angle α. Again, assuming the bunches are short,

$$
\begin{aligned}
L &= L_0 \cdot \frac{1}{\sqrt{\pi}\sigma_z}\int_{-\infty}^{\infty} e^{-\alpha^2 z^2/4\sigma_0^2} e^{-z^2/\sigma_z^2}\, dz\\
&= L_0 \cdot \frac{1}{\sqrt{1+(\alpha\sigma_z/2\sigma_0)^2}} \cdot \frac{\sqrt{1+(\alpha\sigma_z/2\sigma_0)^2}}{\sqrt{\pi}\sigma_z}\int_{-\infty}^{\infty} e^{-[1+(\alpha\sigma_z/2\sigma_0)^2]z^2/\sigma_z^2} dz\\
&= L_0 \cdot \frac{1}{\sqrt{1+(\alpha\sigma_z/2\sigma_0)^2}} \cdot \frac{1}{\sqrt{\pi} z_{rms}}\int_{-\infty}^{\infty} e^{-z^2/z_{rms}^2} dz\\
&= \frac{1}{\sqrt{1+(\alpha\sigma_z/2\sigma_0)^2}} \cdot L_0.
\end{aligned}
$$

The luminosity is reduced, as is the luminous region, which here has rms extent

$$z_{rms} = \frac{\sigma_z}{\sqrt{1+(\alpha\sigma_z/2\sigma_0)^2}}.$$

3 Numerical Examples

Using the above relationships, we can look at some simple examples of the effects of crossing angles, offset centers, and optical distortions on the luminosity.

3.1 Hour Glass

Figure 3 shows the reduction in luminosity as a function of bunch length for the design value of β^* = 35 cm. The vertical lines indicate bunch lengths of $\sigma_z = \beta^*$ and $\sigma_z = 2\beta^*$.

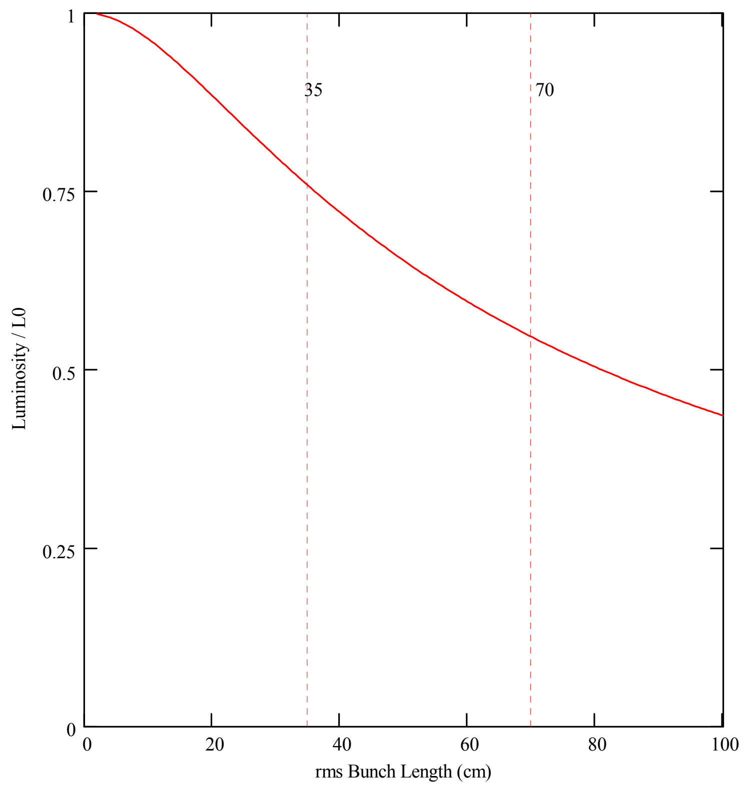

Figure 3: Luminosity as function of σ_z in the Tevatron.

3.2 Beam Separation

Figure 4 shows the effect of an offset between the centers of the two beams, assuming that the two trajectories are parallel. The rms transverse beam size, for $\epsilon = 20\pi$ mm-mrad, and for $\beta^* = 35$ cm, is $\sigma_0 = \sqrt{\epsilon\beta/(6\pi\gamma)} = 33$ μm.

3.3 Crossing Angles and Beam Offsets

Figure 5 shows the reduction in luminosity as a function of crossing angle for typical Tevatron parameters. Here, we use $\beta^* = 35$ cm, $\epsilon = 20\pi$ mm-mrad. For the left graph in Figure 5 we use σ_z = 50 cm. The plot on the right assumes short bunches, and hence no hour glass effect.

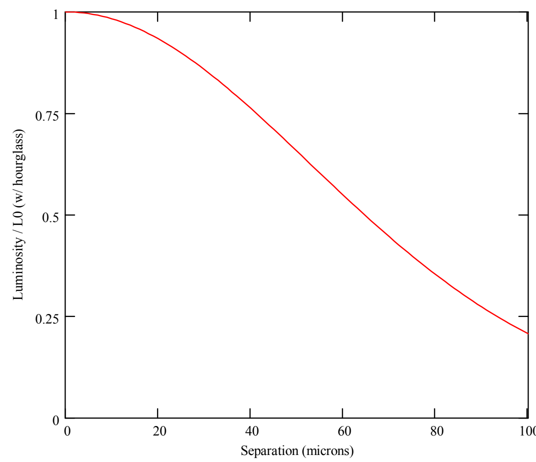

Figure 4: Luminosity as function of beam separation in the Tevatron.

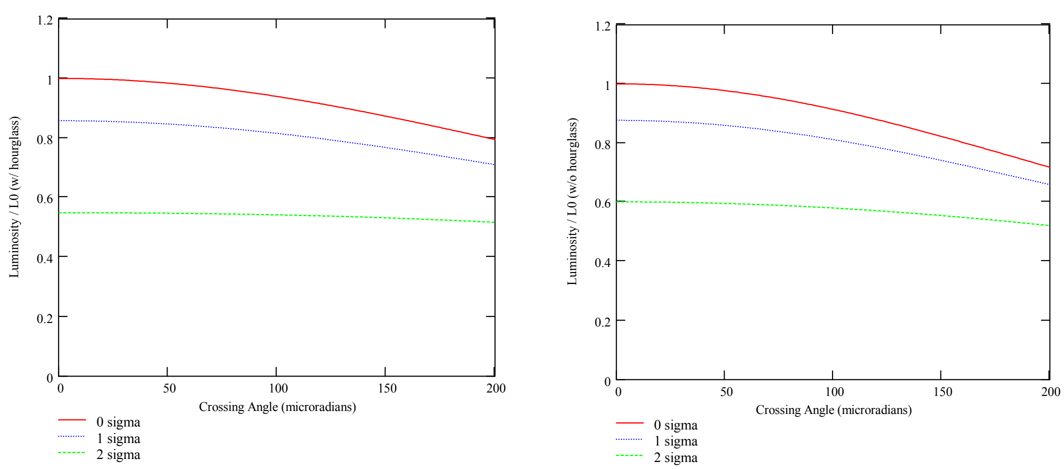

Figure 5: Luminosity as a function of crossing angle in the Tevatron. The three curves, from the top, are for beam separations at the IP of 0, σ_0, and $2\sigma_0$. The right figure assumes short bunches (i.e., with no hour glass effect included). We see that with short bunches, the effect of a crossing angle is slightly enhanced.

3.4 α^*

Even though the minimum amplitude function may be of the right magnitude, the optical focus may occur displaced from the IP by a distance Δz^*. (And, naturally, the focal points for the horizontal and vertical amplitude functions may occur at different locations as well.) The design is for the minimum to occur at $z = 0$, and thus the parameter $\alpha \equiv -(d\beta/dz)/2 = 0$ by design at $z = 0$ as well. If the focus occurs at $z = \Delta z^*$, then the value of α at the IP would be $\alpha* = \Delta z^*/\beta^*$. Optimizing α^* to zero (or, Δz^* to zero) is often referred to operationally as performing an "α-bump."

Figure 6 shows the reduction of luminosity produced by having the focal points of the horizontal and vertical amplitude functions separated by an amount Δz^*, with the minima centered about the ideal IP. For the calculation, the minimum amplitude functions still have values of $\beta^* = 35$ cm, and the bunch length is 50 cm.

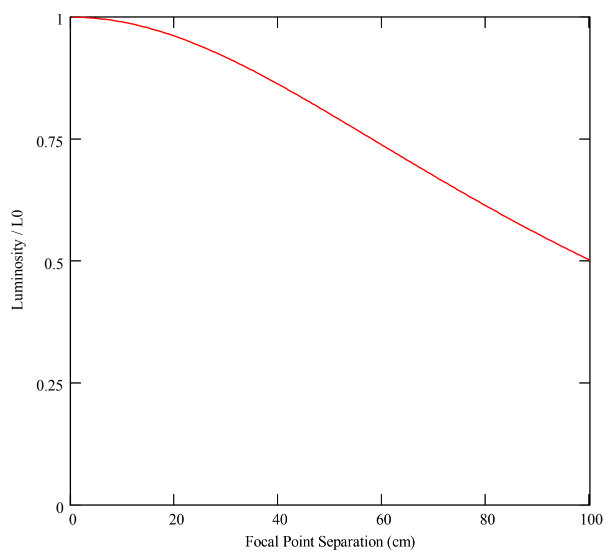

Figure 6: Luminosity reduction due to separation of horizontal and vertical focal points.

3.5 A Combination of Effects

As an example of combining everything together, let's put in some numbers consistent with Run II experience. In the past (though things are better now), optical minima have been observed to be off by as much as ±5-15 cm. So, let's take Δz_x =-5 cm, and $\Delta z_y = +15$ cm. Minimum values of β have been estimated to be 30-50 cm in the past. So, let's take $\beta^*_x = 40$ cm, and $\beta^*_y = 50$ cm. Offsets at the IP and crossing angles can be scanned to optimize luminosity, and so are thought to be relatively small. Suppose Δ_x and Δ_y are 10 μm (1/3 σ_0), and the horizontal and vertical crossing angles are ± 20 μrad. (Note: the least significant bit on the Tevatron BPM system is 150 μm, and the two Collision Point Monitor BPM's are 15 m apart; thus there could be an uncertainty in the angle (separators on minus separators off) of about $\alpha = 300$ μm/15 m = 20 μrad, as an example.)

Using $\sigma_z = 50$ cm, putting everything into Equation 1, and comparing with the ideal luminosity (including hour glass) we get a reduction in luminosity of 18%.

4 Time Evolution of Luminosity

During a store, the luminosity will evolve primarily due to two effects. First, particles are lost due to interactions (that is the purpose, after all!) as well as other beam loss mechanisms and, second, parameters (particularly the transverse and longitudinal emittances) will evolve with time due to various diffusive mechanisms, for example. Thus, if two Interaction Regions have slightly different optics and slightly different crossing angles, etc., then it would not be surprising for their luminosities to evolve differently with time. As the transverse emittance grows, for example, then the implications of separated centroids will diminsh with time. Likewise, hour glass effects, due to bunch length, will change as the longitudinal emittance grows. Recent stores show that the ratio of the luminosity reported by the CDF detector to the luminosity reported by the D0 detector evolves from values of 1.15-1.2 to values of 1.05-1.1 during the course of a single store.

Measurements at the beginning and end of many recent stores show that the transverse emittance of the proton beam evolves from about 20π mm-mrad at the beginning to about 40π m-mrad at the end. The longitudinal emittance grows from about 4 eV-sec to about 7.5 eV-sec during a store.

As an example, let's consider that one Interaction Region (IP #1) in the Tevatron has "ideal" optics and trajectories, while a second IR (IP #2) has the parameters as provided at the end of the last section. Then, the initial luminosity at IP #1 should be about 22% higher than the luminosity at IP #2. However, as the emittances double, then the beam size and and the bunch length will become about 40% larger and their effects on luminosity will decrease. Including only luminosity evolution due to particle interactions (assuming a 60 mbarn cross section) at the two IP's and linear emittance growth over time (which is not exactly what takes place), the resulting development of luminosity at the two IP's is shown in Figure 7. The time evolution of their ratio is depicted in Figure 8.

Several other effects have not been included in this report, for example, the fact that the emittances of the two beams may be different (and are, for the Tevatron), and so on. But hopefully the technique for computing these and other effects on luminosity has been illuminating for the reader.

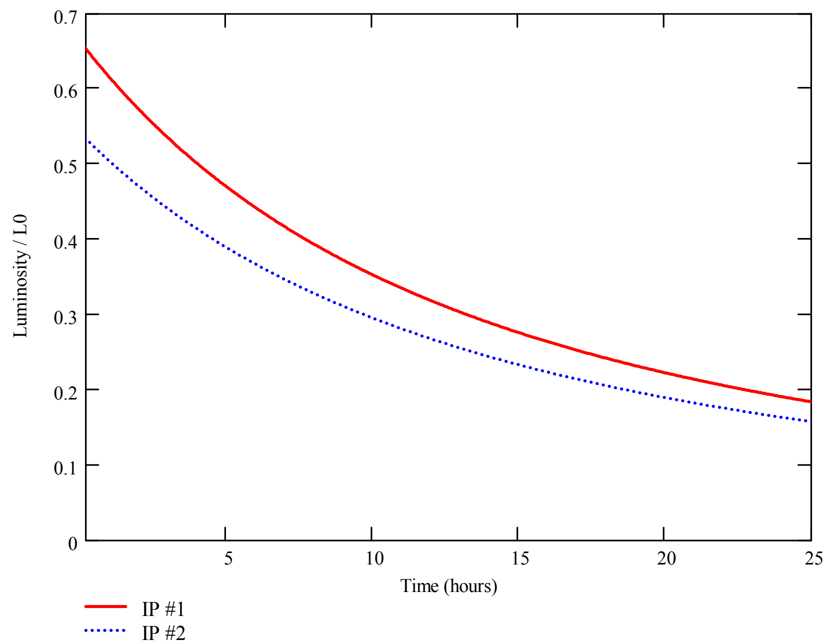

Figure 7: Luminosity during a store for two interaction points. The first IP has "ideal" optics and trajectories (though the hour glass effect is present), while the second IP has the parameters given at the end of Section 3.5.

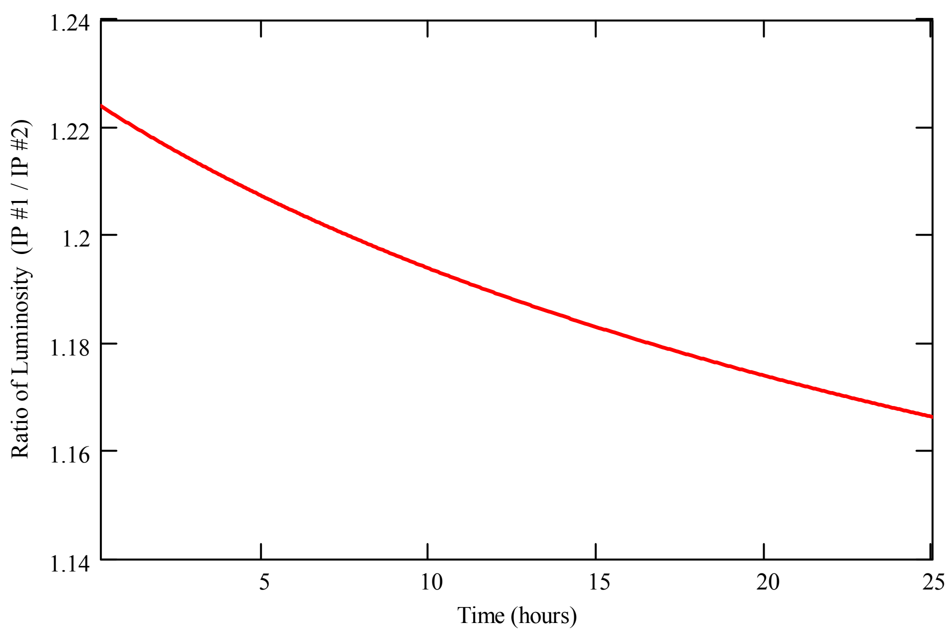

Figure 8: Time evolution of the ratio of the two luminosities of IP #1 to IP #2, shown in Figure 7.